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Large N phase transitions and multi-critical behaviour in generalized 2D QCD

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Abstract.

Using matrix model techniques we investigate the large N limit of generalized $2D$ Yang-Mills theory. The model has a very rich phase structure. It exhibits multi-critical behavior and reveals a third order phase transitions at all genera besides *torus*. This is to be contrasted with ordinary $2D$ Yang-Mills which, at large N, exhibits phase transition only for spherical topology.

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Recently, generalized $2D$ pure gauge SU(N) and U(N) models (gYM$_2$) have been proposed [1] and solved at arbitrary N [2]. The solution generalizes the one obtained for ordinary $2D$ Yang-Mills model (YM$_2$) [3]-[5].

In particular, the partition function for a closed surface of genus g and area A (coupling constant absorbed into area) has the form:

$$Z_g(A) = \sum_r d_r^{2(1-g)} \exp\Big(- \frac{A}{2N}\Lambda(r)\Big) \tag{1}$$

Here, r is an irreducible representation of the gauge group, which is parametrized by its highest weight components, $n_1 \geq ... \geq n_N$, associated with the lengths of lines in the Young table; d_r is its dimension,

$$d_r = \prod_{1\leq i<j\leq N} \Big(1 + \frac{n_i - n_j}{j-i}\Big) \tag{2}$$

and

$$\Lambda(r) = N \sum_{k>0} \frac{a_k}{N^k} C_k(r) \tag{3}$$

with $C_k(r)$ being the k-th Casimir operator eigenvalue,

$$C_k(r) = \sum_{i=1}^{N} l_i^k \gamma_i \quad , \tag{4}$$

where

$$\gamma_i = \prod_{j\neq i} \Big(1 - \frac{1}{l_i - l_j}\Big) \qquad\qquad l_i = n_i + N - i \; . \tag{5}$$

Formula (1) generalizes the known result [4],[5] for YM$_2$ which corresponds to $\Lambda(r)$ with only the $k=2$ term present. The eigenvalue of the second Casimir operator is

$$C_2(r) = \sum_{i=1}^{N} n_i(n_i + N + 1 - 2i) \quad . \tag{6}$$

A generalization of the result for Wilson loop averages [4] is,

$$W_g(C) = \sum_{r_1,\ldots,r_m} \Phi_{r_1\ldots r_m} \prod_{k=1}^{m} d_{r_k}^{2(1-g_k)} \exp\Big(- \frac{A_k}{2N}\Lambda(r_k)\Big) \quad , \tag{7}$$

where m is the number of windows, A_k's are the corresponding areas, g_k is the "genus per window" and the coefficients $\Phi_{r_1\ldots r_m}$ are the U(N) (SU(N)) group factors which depend on the contour topology (see Ref.[4] for details). The result depicted in (7) is obtained from the YM$_2$ result by the replacement $C_2(r) \rightarrow \Lambda(r)$

In Ref. [2] the large N expansion of the generalized action was carried out and a stringy description was derived. The partition function (1) can be written as a sum of $2D$ string maps, where maps with branch points of degree higher than one as well as "microscopic surfaces" play an important role. This generalizes the string interpretation of YM$_2$ given previously by Gross and Taylor [6]-[8]. It is still a challenging problem to find the string action which gives rise to the sum of maps that reproduces the gauge theory partition function. Recently, an important progress toward this goal has been made [9],[10]. The

results of Ref. [10] reveal the topological aspects of the underlying string theory associated with holomorphic maps.

In the present letter we continue to study the gYM_2 theory. We employ the large N approach, elaborated in details in [11], which is well known within the context of matrix models. We demonstrate that gYM_2 exhibits a much richer structure than YM_2. In particular, besides the phase transition realized [12] in YM_2 on a sphere, there are phase transitions at any genus in gYM_2. Moreover, at any genus the model accommodates multi-critical behaviour.

Following [11] we write the partition function (1) at large N as a path integral over continuous Young tables. Namely, we introduce the continuous function,

$$h(x) = \lim_{N\to\infty} \frac{1}{N}\Big(k - \frac{N}{2} - n_k\Big) \quad , \qquad x = \frac{k}{N} \ . \tag{8}$$

Then, the partition function takes the form:

$$Z = \int \prod_{0\le x\le 1} dh(x) e^{S[h(x)]} \quad , \tag{9}$$

$$S = \frac{N^2}{2}\int_0^1 dx \left\{ -AV[h(x)] + (2-2g) \ \fint_0^1 dy \log\Big|h(x)-h(y)\Big| \right\} , \tag{10}$$

where

$$V(h) = \sum_{k>0} \alpha_k h^k \quad .$$

It is implied that the path integral (9) should be taken over $h(x)$'s which satisfy the constraint

$$\frac{dh}{dx} \ge 1$$

which is reminiscence of the dominance condition for group representations.

Then, the saddle point equation is

$$\frac{1}{2}\xi V_h'[h(x)] = \fint \frac{dy}{h(x)-h(y)} \ , \qquad \xi = \frac{A}{2-2g} \tag{11}$$

(ξ^{-1} is the surface density of Euler characteristics).

Introducing the density

$$\rho(h) = \frac{dx}{dh}$$

which should be positive and normalized to

$$\int_a^b dh \ \rho(h) = 1 \ , \tag{12}$$

we rewrite (11) as

$$\frac{1}{2}\xi V'(h) = \fint_a^b \frac{dx\rho(x)}{h-x} \ , \qquad a \le h \le b \ . \tag{13}$$

The constraint $h'(x) \ge 1$ now takes the form

$$\rho(h) \le 1 \quad . \tag{14}$$

The solution of (13) is

$$\rho(h) = \frac{1}{\pi} P(h) \sqrt{(h-a)(b-h)} \ , \tag{15}$$

where the polynomial $P(h)$ and the values a and b are uniquely defined by the $h \to \infty$ asymptotic behavior

$$\frac{1}{2}\xi V'(h) - P(h)\sqrt{(h-a)(h-b)} \sim \frac{1}{h} \ . \tag{16}$$

In the YM$_2$ case ($V = h^2$), the solution (constraint (14) is ignored) is:

$$\rho(h) = \frac{\xi}{\pi}\sqrt{b^2 - h^2} \qquad |h| \leq 1 \qquad \xi b^2 = 2 \ . \tag{17}$$

Thus, the solution exists only for genus zero, i.e., the sphere. At $2\xi < \pi^2$ Eq.(14) is satisfied for all h. It means that in sum over representations we can ignore the dominance condition and all n_k's now run independently from $-\infty$ to $+\infty$. In other words the U(N) (as well as SU(N)) model now becomes (effectively) $\bigotimes^N$U(1) model, i.e., abelian. In terms of the original U(N) matrix variables it means that all unitary matrices are frozen near I. This is the weak coupling (small area) phase of the model.

For $2\xi > \pi^2$, there is a region of values of h where solution (17) does not satisfy the inequality (14). In this region the inequality (14) is saturated, $\rho(h) = 1$. Then, the solution of the saddle point equation is [12]:

$$\rho(h) = \begin{cases} -\frac{2}{\pi a h}\sqrt{(a^2-h^2)(h^2-b^2)}\,\Pi_1\Big(-\frac{b^2}{h^2}, \frac{b}{a}\Big) & \text{for } -a < h < -b \\ 1 & \text{for } -b < h < b \\ \frac{2}{\pi a h}\sqrt{(a^2-h^2)(h^2-b^2)}\,\Pi_1\Big(-\frac{b^2}{h^2}, \frac{b}{a}\Big) & \text{for } b < h < a \end{cases} \tag{18}$$

where $\Pi_1(x,k)$ is the complete elliptic integral of the third kind with modulus $k = \frac{b}{a}$ and the parameters are determined by the equations

$$a(2E - k'^2 K) = 1 \qquad\qquad aA = 4K \ , \tag{19}$$

where[2] $k' = \sqrt{1-k^2}$ is the complementary modulus and

$$K = K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad\qquad E = E(k) = \int_0^1 dx \sqrt{\frac{1-k^2x^2}{1-x^2}} \ .$$

At the critical value, $2\xi_c = \pi^2$ ($A_c = \pi^2$), the third order phase transition [14] takes place. It was observed by Douglas and Kazakov [12].

We observe that in the generalized case the constraint (14) can be satisfied also for negative ξ by an appropriate choice of the coupling constants α_i, which implies the existence of phase transitions at higher genera, except the *torus*. (In the latter case the saddle point equation is an algebraic equation for h with solution $h = const$ and, therefore, the constraint $h' \geq 1$ is not satisfied). Moreover, there is a multi-critical behaviour at each genus. Indeed, since the semicircle distribution (17) is now deformed by a polynomial (15), there are several disconnected regions of h where the constraint (14) is satisfied.

[2] All notations are taken from the book [13].

Consider, for example, the simplest generalized case

$$V(h) = h^2 + \frac{2\alpha}{3} h^3$$

which is the well-known φ^3 model. The solution is [15]:

$$\rho(h) = \frac{\xi}{\pi}[1 + \frac{\alpha}{2}(a+b) + \alpha h]\sqrt{(h-a)(b-h)} \quad , \tag{20}$$

and the values of a and b are determined from

$$\begin{aligned} &\tfrac{\alpha}{4}(b-a)^2 + (a+b)[1 + \tfrac{\alpha}{2}(a+b)] = 0 \\ &\xi(b-a)^2[1 + \alpha(a+b)] = 8 \quad , \end{aligned} \tag{21}$$

which leads to

$$\begin{aligned} a &= \tfrac{2z-1}{2\alpha} - 2\sqrt{\tfrac{z}{\xi}} \\ b &= \tfrac{2z-1}{2\alpha} + 2\sqrt{\tfrac{z}{\xi}} \quad , \end{aligned} \tag{22}$$

with

$$z^3 - \frac{1}{4}z + \frac{\alpha^2}{2\xi} = 0 \quad . \tag{23}$$

At negative ξ we should take the $z < 0$ solution which exists for

$$\alpha^2 < -\frac{\xi}{3\sqrt{12}} \quad .$$

The critical value $\xi_c = \xi_c(\alpha)$ should be defined from (20) through the condition

$$\rho_{\max}(h) = 1. \tag{24}$$

It is clear that similar situation prevails for the general case. For $\xi < \xi_c$, where ξ_c is defined by (24) and ρ defined by (15), the weak coupling phase is realized.

In the strong coupling phase ($\xi > \xi_c$) the saddle point equation (13) should be solved in presence of the constraint (14). The only solution is

$$\rho(h) = \begin{cases} 1 & a_k \le h \le b_k \qquad k = 1, ..., m \\ \tilde{\rho}(h) & b_k \le h \le a_{k+1} \qquad k = 0, ..., m \end{cases} \tag{25}$$

where $b_0 = a, \quad a_{m+1} = b$ and m is the number of intervals (a_k, b_k) where

$$\frac{1}{\pi} P(h)\sqrt{(h-a)(b-h)} > 1 \quad .$$

The function $\tilde{\rho}(h)$ has to be defined via the equation

$$\frac{1}{2}\xi V'(h) - \sum_{k=1}^{m} \log \frac{h-b_k}{h-a_k} = \sum_{k=0}^{m} ⨍_{b_k}^{a_{k+1}} \frac{dx\tilde{\rho}(x)}{h-x} \quad . \tag{26}$$

Wilson loop averages can be calculated following the same steps as in the YM_2 case on a sphere [16],[17]. A simple (without self-intersections) contractible [3] loop divides the surface into two "windows" with areas A_1, A_2 and genera per "window" g_1, g_2. The corresponding expectation value can be calculated in a similar way to what was done for YM_2 on a sphere [16]:

$$W(A_1, g_1; A_2, g_2) = \int dh\eta(h) D^{2-2g_2}(h) e^{-\frac{1}{2}A_2 V'(h)} \quad , \tag{27}$$

where the quantities $\eta(h)$ and $D(h)$ are precisely the same as in the spherical case (see Ref.[16]) and are expressed through $\rho(h)$. We note that Eq.(27) is symmetric with respect to the exchange $(A_1, g_1) \to (A_2, g_2)$. Again, as in the spherical case, $W(A_1, 0) = W(0, A_2) = 1$.

Expression (27) is valid in both the weak and the strong coupling phases. A complete information concerning the phase transition is contained in the function $\rho(h)$. This function is given by (15) in the weak coupling phase and by (25)-(26) in the strong coupling phase.

Starting from (27) the complete set of Wilson loop averages can be calculated by means of the loop equation [18] in its simple two-dimensional formulation [19], similarly to what was done for YM_2 on a sphere [17].

Several remarks are in order. First, the non-trivial saddle point for $g > 1$ exists only if the coupling constant corresponding to the highest Casimir eigenvalue in the generalized potential (3) is negative. In terms of the matrix model it implies that the potential $V(h)$ has a negative leading term, i.e., we deal with a matrix model with an *upside-down* potential. It is easy to check that there is no phase transition on a sphere for the case of an upside-down potential. Our second remark concerns the function $\xi_c = \xi_c(g)$. For any particular model (fixed couplings α_k and maximal order of the Casimir operators entering in (3)) the critical value ξ_c is fixed (it is unambiguously defined by the saddle point equation). Therefore, increasing the genus the critical area increases proportionally in order to keep fixed the critical number of handles per area (the density of the Euler characteristics). Also, if we fix the area A of the surface, then with the increase of the genus the number of multi-critical points decreases and for $g > 1 + \frac{A}{2|\xi_c|}$ there is no phase transition.

There are several open questions which follow from our observations. We have presented only a qualitative picture of the critical behavior in gYM_2. It is of interest to get the critical exponents associated with the phase transitions.

In the weak coupling phase, where the constraint (14) can be ignored, the model coincides with the usual (large N) hermitian matrix model with arbitrary potential. The latter has been investigated thoroughly during recent years. It describes (upon fine tuning of the parameters) two-dimensional quantum gravity (coupled to non-unitary CFT matter). It can be also viewed as string embedded in $D = 0$ space. It is very intriguing to find out whether there exists an hermitian matrix model which corresponds to the strong coupling phase of gYM_2. If such a matrix model description is found, then the obvious question is to which quantum gravity string theory it corresponds and what is its relation to the stringy formulation of gYM_2 [2],[6]-[10]. Clearly, the stringy approach to gYM_2 holds

[3] An expectation value of non-contractible loop might be zero due to pure topological reason. A complete set of zero-valued loops was found in [4].

in the strong coupling phase. As a first step toward the construction of such a matrix model one could try to introduce the constraint (14) directly into the action (10) of the model (the constraint can be included in the path integral over $h(x)$ as a step function $\theta(|h'(x)|-1)$ which can be incorporated into the action, for example, via a Lagrange multiplier or through some convenient parametrization).

One can start with the lattice formulation of gYM_2 and integrate systematically over the link gauge variables. This gives rise to an effective action for the auxilliary B field which is again a matrix model. The matrix model formulation can help toward finding an integrable structure of gYM_2. [4]

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